The regularization of a propositional logic

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Abstract

A variety $\mathcal{K}$ is called (strongly) *irregular* whenever it satisfies an identity of the kind $f(x, y) \approx x$, where $f(x, y)$ is any term of the language in which x and y really occur. A variety is *regular*, when it is not irregular. Examples of irregular varieties abound in logic, since every variety with a lattice reduct is irregular as witnessed by the term $f(x, y) \coloneqq x \wedge (y \vee x)$. On the other hand, an identity $\varphi \approx \psi$ is said to be *regular* provided that exactly the same variables occur in φ and ψ. The algebraic study of regular varieties traces back to the pioneering work of Płonka, who introduced a class-operator $\mathcal{P}_l(\cdot)$ nowadays called *Płonka sum*, and used it to prove that any regular variety $\mathcal{K}$ can be represented as Płonka sums of a suitable irregular variety $\mathcal{V}$, in symbols $\mathcal{P}_l(\mathcal{V}) = \mathcal{K}$. In this case $\mathcal{K}$ is called the *regularization* of $\mathcal{V}$.

Over the years, regular varieties have been studied in depth from purely algebraic and categorial perspectives. However, the recent discovery that the regularization of the variety of Boolean algebras is the algebraic semantics of Paraconsistent Weak Kleene logic, i.e. a particular three-valued Kleene-like logic, showed that the notion of regularization could find interesting applications in logic as well.

In the first part of this talk, we introduce Paraconsistent Weak Kleene logic, as a case study of a *regular* logic and show its relationships with its algebraic semantics.

Taking advantage of the concrete example, in the second part of the talk, we develop the notion of *logic-based* regularization, which on the one hand extends and subsumes the known algebraic one, and on the other hand explains on general grounds the relations between classical logic and Paraconsistent Weak Kleene logic. Our investigation is carried on in the framework of abstract algebraic logic.

In order to introduce the notion of regularization of a given logic, we need to extend first the construction of Płonka sums to logical matrices. A *direct system* of logical matrices is a triple $X = \langle I, \{\mathbf{A}_i, F_i\}_{i\in I}, \varphi_{ij}\rangle$ where, I is a join-semilattice of indexes, $\langle \mathbf{A}_i, F_i\rangle_{i\in I}$ is a family of logical matrices, and $\varphi_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ is a homomorphism such that $\varphi_{ij}[F_i] \subseteq F_j$ for every $i \leq j$. The Płonka sum over a direct system X is the logical matrix $\mathcal{P}_l(X) \coloneqq \langle \mathcal{P}_l(\mathbf{A}_i), \bigcup_{i\in I} F_i\rangle$. Now, recall that every logic $\vdash$ can be naturally associated with a class of logical matrices, usually called the *reduced models* of $\vdash$. With this technology at hand, we are ready to define the *regularization* $\vdash_r$ of a logic $\vdash$ as the logic determined by all Płonka sums of reduced models of $\vdash$. Remarkably, the regularization $\vdash_r$ could be equivalently defined syntactically, since it is not difficult to observe that

$$\Gamma \vdash_r \varphi \iff \text{ there is } \Delta \subseteq \Gamma \text{ s.t. } \mathrm{var}(\Delta) \subseteq \mathrm{var}(\varphi) \text{ and } \Delta \vdash \varphi.$$

In other words, regular logics admit also a syntactical definition as logics of variable inclusion. From this starting point, we investigate logico-algebraic properties of $\vdash_r$ such as the Hilbert-style calculi associated, the structure of its reduced models, its location in the Leibniz and Frege hierarchy (both as a logic and as a Gentzen system).